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Critical dynamics of the Gaussian model with multispin transitions

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In this paper, we present a multispin transition mechanism, which is an extension of the Glauber one, to investigate critical dynamics. By exactly solving the master equation, the influence of the multispin transition mechanism on the dynamic critical behavior is studied for the Gaussian model with nearest-neighbor interactions on d-dimensional lattices ($d=1$, 2, and 3). The time evolution of magnetization is exactly calculated, and the exact results of relaxation time and dynamic critical exponent are obtained. Our models are divided into two kinds: one is the spin-cluster transition and the other is the arbitrary multispin transition. It is found that there are different relaxation times, but the same dynamical critical exponent for different kinds of multispin transitions. The results show that the dynamical critical exponents are independent of spatial dimensions and configurations of transitional spins, and that the dynamical critical exponent is the same as that of the Glauber dynamics, and thus give a strong support to the simple single-spin-transition dynamics. Finally, we give a brief discussion on the results.

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I. INTRODUCTION

In the field of the nonequilibrium statistical mechanics and critical dynamics, as a soluble problem the one-dimensional kinetic Ising model was originally studied by Glauber [1] and Kawasaki [2] in 1963 and 1966, respectively. In the pioneering works, they consider that the spin system relaxes by a series of single-spin flips, named the Glauber mechanism, and by a series of spin-pair exchanges, named the Kawasaki mechanism, respectively. Since then, many works in this field have been done and many developments have occurred [3–12]. Among these works, two important aspects are to investigate the influence of spin-transition mechanism and spin-transition rate on the dynamic critical properties of a system. For the first aspect, in addition to the single-spin-flip and spin-pair exchange models, some works have been done. For example, Felderhof, Suzuki, and Hilhorst studied particular multiple-spin-flip models for the one-dimensional Ising system and exactly found that the system has the dynamical exponent $z=2$, if the spin-flip rates is the same as Glauber's choice [13]. Recently, Zhu and Yang generalized Glauber's critical dynamics; they considered a single-spin transition instead of a single-spin flip, so that it is appropriate for both discrete and continuous spin models [14,15]. In addition, Zhu and Zhu also studied the Kawasaki dynamics with spin-pair exchange for the Gaussian model as well [16]. For the second aspect, some works show that the choice of the transition rate influences the dynamic critical properties. For example, using the renormalization group approach, Deker and Haake studied a kinetic Ising chain and found that spin-flip rates different from Glauber's choice may result in different exponents z. Haake and Thol also studied double-spin-flip systems and found that they have different dynamical exponents, if the flip rates are different [17].

In this paper, we present a multispin transition mechanism to investigate critical dynamics. This mechanism is more real, but more complex mathematically. To make the analytical solution possible, we employ the Gaussian model. Our results show that the dynamical critical exponent is the same as that of the Glauber dynamics, and thus give a strong support to the simple single-spin-transition dynamics (the Glauber dynamics).

As is well known, in the study of the dynamic critical phenomena, the main task is to calculate the time evolution of the local order parameter and critical dynamical exponent, and the key step is to determine the transition rate of the spins. In this paper, we assume that the multispin transition rate is proportional to $\exp(\mathcal{H})$ under the requirement of the detailed balance condition, where $\mathcal{H}$ is the effective Hamiltonian related to transitional spins. In this case, we can exactly solve the master equation of the Gaussian model. Our investigation finds that the dynamic critical exponent $z=2$ not only for different dimensional lattices but also for different kinds (see below) of multispin transitions, which is in accord with the result of Ref. [14], in which only the single-spin transition is considered.

This paper is organized as follows. In Sec. II, we give a generalized dynamic version of the spin models with multispin transitions. In Sec. III, we solve exactly the one-dimensional kinetic Gaussian model with double-spin transitions. Section IV is a two-dimensional case, involving five-spin transitions and four-spin transitions. Section V gives the results of three-dimensional and d-dimensional lattices, and Sec. VI gives a brief conclusion and discussion. Some of the more tedious calculations of Secs. III, IV, and V are given in the Appendixes.

II. MASTER EQUATION OF MULTISPIN TRANSITIONS

In the kinetic spin model, originally proposed by Glauber for the one-dimensional Ising system, the probability distri-

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bution of the spin configuration evolution is described by the master equation [1], in which only the single-spin flip is allowed each time while the others remain momentarily fixed. As we have mentioned above, at present we study the generalized case, in which multispin transitions can occur each time and the spin values can be discrete or continuous. In our multispin-transition mechanism, the transitional spins can be concentrated and neighboring each other which is called a spin cluster, or distributed in spatial distribution. First, in this section we derive the general form of the master equation of spin-cluster transitions. The distributed case is studied in the Sec. III.

Consider a spin system on a simple cubic lattice with N sites. There is a spin σ_j on the jth site. Let us label a site cluster by c_i containing n lattice sites. The corresponding spin cluster is $\sigma_{c_i}\equiv(\sigma_i^1,\sigma_i^2,\dots,\sigma_i^k,\dots,\sigma_i^n)$, where superscript k denotes the kth spin in the spin cluster. We assume that all spins in the cluster can change from σ_{c_i} to $\hat{\sigma}_{c_i}$ simultaneously, while the others remain momentarily fixed, where $\hat{\sigma}_{c_i}=(\hat{\sigma}_i^1,\hat{\sigma}_i^2,\dots,\hat{\sigma}_i^k,\dots,\hat{\sigma}_i^n)$. Similar to the Glauber dynamics, we describe the time-dependent behavior of the nonequilibrium probability distribution $P(\sigma,t)$ by means of a master equation. Considering that every spin cluster can change, the master equation is written as

$$\frac{d}{dt}P(\sigma,t)=\frac{1}{n}\sum_{c_i}\sum_{\hat{\sigma}_{c_i}}[-W(\sigma_{c_i}\rightarrow\hat{\sigma}_{c_i})P(\sigma,t)+W(\hat{\sigma}_{c_i}\rightarrow\sigma_{c_i})P(\sigma_1,\dots,\hat{\sigma}_{c_i},\dots,\sigma_N,t)], \tag{1}$$

where $W(\sigma_{c_i}\rightarrow\hat{\sigma}_{c_i})$ is the transition rate from spin configuration $\sigma\equiv(\sigma_1,\sigma_2,\dots,\sigma_{c_i},\dots,\sigma_N)$ to $(\sigma_1,\sigma_2,\dots,\hat{\sigma}_{c_i},\dots,\sigma_N)$, the summation $\Sigma_{\hat{\sigma}_{c_i}}$ is carried out over all possible transitions of the spin cluster σ_{c_i}, and Σ_{c_i} goes over all clusters in the system, the coefficient $1/n$ comes from the fact that each spin belongs to n spin clusters (see Sec. III A and Figs. 2 and 3). $W(\sigma_{c_i}\rightarrow\hat{\sigma}_{c_i})$ satisfies the normalized condition

$$\sum_{\hat{\sigma}_{c_i}}W(\sigma_{c_i}\rightarrow\hat{\sigma}_{c_i})=\frac{1}{\tau_s}, \tag{2}$$

where τ_s defines the unit of time. We assume that $W(\sigma_{c_i}\rightarrow\hat{\sigma}_{c_i})$ and $W(\hat{\sigma}_{c_i}\rightarrow\sigma_{c_i})$ satisfy the detailed balance condition [1,14]

$$W(\sigma_{c_i}\rightarrow\hat{\sigma}_{c_i})P_{\mathrm{eq}}(\sigma)=W(\hat{\sigma}_{c_i}\rightarrow\sigma_{c_i})P_{\mathrm{eq}}(\sigma_1,\dots,\hat{\sigma}_{c_i},\dots,\sigma_N), \tag{3}$$

where $P_{\mathrm{eq}}(\sigma)\equiv P_{\mathrm{eq}}(\sigma_1,\dots,\sigma_{c_i},\dots,\sigma_N)$ is the equilibrium Boltzmann distribution function $P_{\mathrm{eq}}(\sigma)=(1/Z)\exp[-\beta H(\sigma)]$, Z is the partition function, and $H(\sigma)$ is the Hamiltonian of the system. $\beta=1/(k_BT)$, k_B is Boltzmann's constant, and T is the absolute temperature. The transition rate can be written as

$$\tau_s W(\sigma_{c_i}\rightarrow\hat{\sigma}_{c_i})=\frac{1}{Q_{c_i}}\exp[\mathcal{H}_{c_i}], \tag{4}$$

where

$$Q_{c_i}=\sum_{\hat{\sigma}_{c_i}}\exp[\mathcal{H}_{c_i}] \tag{5}$$

is the normalized factor and $\mathcal{H}_{c_i}=-\beta H_{c_i}(\hat{\sigma}_{c_i},\{\sigma_{j\notin c_i}\})$ is the Hamiltonian related to spin cluster $\hat{\sigma}_{c_i}$. It is easy to prove that expression (4) satisfies conditions (2) and (3).

For studying the dynamic critical phenomena, we are interested in the time evolution of the local magnetization $s_k(t)$, which is the ensemble average value of $\sigma_k(t)$, i.e.,

$$s_k(t)=\sum_{\sigma}\sigma_kP(\sigma,t). \tag{6}$$

Let σ_k multiply both sides of Eq. (1) and sum over all values of the variable σ, we get

$$\frac{d}{dt}\sum_{\sigma}\sigma_kP(\sigma,t)=\frac{1}{n}\sum_{\sigma}\sigma_k\left(\sum_{c_i(k\notin c_i)}+\sum_{c_i(k\in c_i)}\right)\times\sum_{\hat{\sigma}_{c_i}}[-W(\sigma_{c_i}\rightarrow\hat{\sigma}_{c_i})P(\sigma,t)+W(\hat{\sigma}_{c_i}\rightarrow\sigma_{c_i})P(\dots,\hat{\sigma}_{c_i},\dots,t)],$$

where the summation $\Sigma_{c_i(k\in c_i)}$ is carried out over the clusters containing site k and $\Sigma_{c_i(k\notin c_i)}$ over the others. We first consider the summation for $c_i(k\notin c_i)$. Because of $\sigma=(\sigma_{c_i},\{\sigma_{j\notin c_i}\})$, we have

$$\sum_{\sigma}\sigma_k\sum_{c_i(k\notin c_i)}\sum_{\hat{\sigma}_{c_i}}[-W(\sigma_{c_i}\rightarrow\hat{\sigma}_{c_i})P(\sigma,t)+W(\hat{\sigma}_{c_i}\rightarrow\sigma_{c_i})P(\dots,\hat{\sigma}_{c_i},\dots,t)]=\sum_{\{\sigma_{j\notin c_i}\}}\sigma_k\sum_{c_i(k\notin c_i)}\sum_{\sigma_{c_i},\hat{\sigma}_{c_i}}[-W(\sigma_{c_i}\rightarrow\hat{\sigma}_{c_i})P(\sigma,t)+W(\hat{\sigma}_{c_i}\rightarrow\sigma_{c_i})P(\dots,\hat{\sigma}_{c_i},\dots,t)]. \tag{7}$$

Note that

$$\sum_{\sigma_{c_i},\hat{\sigma}_{c_i}} W(\hat{\sigma}_{c_i}\to\sigma_{c_i})P(\ldots,\hat{\sigma}_{c_i},\ldots,t)=\sum_{\sigma_{c_i},\hat{\sigma}_{c_i}} W(\sigma_{c_i}\to\hat{\sigma}_{c_i})P(\ldots,\sigma_{c_i},\ldots,t).$$

Expression (7) is thus equal to zero. Moreover, since there are n different clusters $c_i(c_i\supset k)$, we obtain

$$\sum_{\sigma}\sigma_k\sum_{c_i(k\in c_i)}\sum_{\hat{\sigma}_{c_i}}[-W(\sigma_{c_i}\to\hat{\sigma}_{c_i})P(\sigma,t)+W(\hat{\sigma}_{c_i}\to\sigma_{c_i})P(\ldots,\hat{\sigma}_{c_i},\ldots,t)]=\sum_{\sigma}\sigma_k\sum_{i=1}^{n}\sum_{\hat{\sigma}_{c_i}}[-W(\sigma_{c_i}\to\hat{\sigma}_{c_i})P(\sigma,t)$$
$$+W(\hat{\sigma}_{c_i}\to\sigma_{c_i})P(\ldots,\hat{\sigma}_{c_i},\ldots,t)]=-\frac{n}{\tau_s}s_k+\sum_{\sigma}\sum_{i=1}^{n}\sum_{\hat{\sigma}_{c_i}}\hat{\sigma}_kW(\sigma_{c_i}\to\hat{\sigma}_{c_i})P(\sigma,t).$$

Therefore, the differential equation of the local magnetization is written as

$$\frac{d}{dt}s_k(t)=-\frac{s_k(t)}{\tau_s}+\frac{1}{n}\sum_{\sigma}\sum_{i=1}^{n}\sum_{\hat{\sigma}_{c_i}}\hat{\sigma}_kW(\sigma_{c_i}\to\hat{\sigma}_{c_i})P(\sigma,t), \tag{8}$$

which is our starting point in studying dynamic critical phenomena. Obviously, the differential equation can be used for all spin models, in principle. However, because of its complexity it cannot be exactly solved, in general. In this paper, we employ the Gaussian model with nearest-neighbor (NN) interactions. Fortunately, the kinetic Gaussian model can be exactly solved.

The Gaussian model was proposed by Berlin and Kac in 1952 [18]. The Hamiltonian of a Gaussian spin system can be written as

$$-\beta H=K\sum_{\langle ij\rangle}\sigma_i\sigma_j, \tag{9}$$

where the Gaussian spin σ_i can take any real values in the range of $(-\infty,\infty)$, $K=J/(k_BT)$, J is the NN exchange integral, and the sum $\Sigma_{\langle i,j\rangle}$ runs over all NN spin pairs. The Gaussian probability distribution for σ_i is [14,18]

$$f(\sigma_i)d\sigma_i=\sqrt{\frac{b}{2\pi}}\exp\left(-\frac{b}{2}\sigma_i^2\right)d\sigma_i, \tag{10}$$

which denotes the probability of finding a given spin σ_i between σ_i and $\sigma_i+d\sigma_i$, where b is the Gaussian distribution constant, $b>0$.

III. ONE-DIMENSIONAL KINETIC GAUSSIAN MODEL

A. Double-spin-cluster transitions

For explicitness, in this section we study the one-dimensional kinetic Gaussian system with double-spin transitions. First, we study the case of double-spin-cluster transitions. Consider a spin cluster $\sigma_{c_i}=(\sigma_i,\sigma_{i+1})$ that can change from (σ_i,σ_{i+1}) to $\hat{\sigma}_{c_i}=(\hat{\sigma}_i,\hat{\sigma}_{i+1})$ simultaneously. Based on Eq. (8), we only need to consider two clusters: $c_1=(k,k+1)$ and $c_2=(k-1,k)$. We thus write the differential equation as

$$\frac{d}{dt}s_k(t)=-\frac{1}{\tau_s}s_k(t)+\frac{1}{2}\sum_{\sigma}\sum_{\hat{\sigma}_{c_1}}\hat{\sigma}_kW(\sigma_{c_1}\to\hat{\sigma}_{c_1})P(\sigma,t)$$
$$+\frac{1}{2}\sum_{\sigma}\sum_{\hat{\sigma}_{c_2}}\hat{\sigma}_kW(\sigma_{c_2}\to\hat{\sigma}_{c_2})P(\sigma,t). \tag{11}$$

In order to solve Eq. (11), we ought to calculate the summations $\Sigma_{\hat{\sigma}_{c_i}}\hat{\sigma}_kW(\sigma_{c_i}\to\hat{\sigma}_{c_i})$, $i=1,2$. According to expression (9), the Hamiltonians related to spin clusters $\sigma_{c_1}=(\sigma_k,\sigma_{k+1})$ and $\sigma_{c_2}=(\sigma_{k-1},\sigma_k)$ are written as

$$\mathcal{H}_{c_1}=-\beta H_{k,k+1}(\hat{\sigma}_k,\hat{\sigma}_{k+1},\{\sigma_{j\neq k,k+1}\})$$
$$=K(\hat{\sigma}_k\sigma_{k-1}+\hat{\sigma}_k\hat{\sigma}_{k+1}+\hat{\sigma}_{k+1}\sigma_{k+2}) \tag{12}$$

and

$$\mathcal{H}_{c_2}=-\beta H_{k-1,k}(\hat{\sigma}_{k-1},\hat{\sigma}_k,\{\sigma_{j\neq k-1,k}\})$$
$$=K(\hat{\sigma}_{k-1}\sigma_{k-2}+\hat{\sigma}_{k-1}\hat{\sigma}_k+\hat{\sigma}_k\sigma_{k+1}), \tag{13}$$

respectively. Because the spins are continuous in the Gaussian model, we can change the summation $\Sigma_{\hat{\sigma}_k}(\cdots)$ into integral $\int_{-\infty}^{+\infty}d\hat{\sigma}_kf(\hat{\sigma}_k)(\cdots)$, where $f(\hat{\sigma}_k)$ is given by expression (10). Based on Eqs. (5) and (12), the normalized factor Q_{c_1} is written as

$$Q_{c_1}=\frac{b}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}d\hat{\sigma}_kd\hat{\sigma}_{k+1}\exp\left[\mathcal{H}_{c_1}-\frac{b}{2}(\hat{\sigma}_k^2+\hat{\sigma}_{k+1}^2)\right]$$
$$=\sqrt{\frac{b}{2\pi}}\exp\left(\frac{K^2}{2b}\sigma_{k+2}^2\right)\int_{-\infty}^{+\infty}d\hat{\sigma}_k\exp\left[K\left(\sigma_{k-1}\right.\right.$$
$$\left.\left.+\frac{K}{b}\sigma_{k+2}\right)\hat{\sigma}_k-\frac{b}{2}\left(1-\frac{K^2}{b^2}\right)\hat{\sigma}_k^2\right]. \tag{14}$$

In terms of Eqs. (4), (14), and the following expression:

$$\int_{-\infty}^{+\infty} x\, dx \exp\left(ax - \frac{\lambda}{2} x^2 \right) = \frac{a}{\lambda} \int_{-\infty}^{+\infty} dx \exp\left(ax - \frac{\lambda}{2} x^2 \right) \quad (\lambda > 0),$$

we can obtain

$$\tau_s \sum_{\hat{\sigma}_{c_1}} \hat{\sigma}_k W(\sigma_{c_1} \to \hat{\sigma}_{c_1}) = \tau_s \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \hat{\sigma}_k W(\sigma_{c_1} \to \hat{\sigma}_{c_1}) \times f(\hat{\sigma}_k, \hat{\sigma}_{k+1}) d\hat{\sigma}_k d\hat{\sigma}_{k+1}$$
$$= \frac{1}{Q_{c_1}} \sqrt{\frac{b}{2\pi}} \exp\left(\frac{K^2}{2b} \sigma_{k+2}^2 \right) \times \int_{-\infty}^{+\infty} \hat{\sigma}_k d\hat{\sigma}_k \exp\left[K\left(\sigma_{k-1} + \frac{K}{b} \sigma_{k+2} \right) \hat{\sigma}_k - \frac{b}{2}\left(1 - \frac{K^2}{b^2} \right) \hat{\sigma}_k^2 \right]$$
$$= \frac{K(b\sigma_{k-1} + K\sigma_{k+2})}{b^2 - K^2}. \tag{15}$$

For the cluster c_2, we can also obtain

$$\tau_s \sum_{\hat{\sigma}_{c_2}} \hat{\sigma}_k W(\sigma_{c_2} \to \hat{\sigma}_{c_2}) = \frac{K(b\sigma_{k+1} + K\sigma_{k-2})}{b^2 - K^2}. \tag{16}$$

Thus, from Eqs. (15), (16), and (11) we obtain the differential equation of $s_k(t)$ as

$$\tau_s \frac{d}{dt} s_k(t) = -s_k(t) + \frac{K[b(s_{k-1} + s_{k+1}) + K(s_{k-2} + s_{k+2})]}{2(b^2 - K^2)}. \tag{17}$$

Equation (17) can be exactly solved by defining a generating function. The detailed process is given in Appendix A. Here, we give the solution as follows:

$$s_k(t) = e^{-t/\tau_s} \sum_{m,n=-\infty}^{\infty} s_l(0) I_m(x_1) I_n(x_2), \quad l = k - (m + 2n), \tag{18}$$

where $I_m(x)$ is the Bessel function, both m and n are integers, and

$$x_1 = \frac{Kb}{b^2 - K^2} \frac{t}{\tau_s}, \quad x_2 = \frac{K^2}{b^2 - K^2} \frac{t}{\tau_s}. \tag{19}$$

We note that solution (18) is very complex. In order to get the explicit relation of $s_k(t)$ with time t, we study the asymptotic behavior of $s_k(t)$ at $t \to \infty$. Using asymptotic expression of the first kind of imaginary argument Bessel function [14], we obtain the time evolution of the local magnetization

$$s_k(t) \sim e^{-t/\tau_s} \sum_{m,n=-\infty}^{\infty} s_l(0) \frac{e^{x_1 + x_2}}{\sqrt{x_1 x_2}} = \frac{b^2 - K^2}{K\sqrt{bKt}} \tau_s \sum_{m,n=-\infty}^{\infty} s_l(0) e^{-t/\tau}, \tag{20}$$

where $s_l(0)$ is the initial value of $s_l(t)$, and τ is the relaxation time given by

$$\tau = \frac{b - K}{b - 2K} \tau_s. \tag{21}$$

From the above expression, we can see that the relaxation time only depends on the temperature of the system. For the Gaussian model, we know that the critical point is determined by

$$K_c = \frac{J}{k_B T_c} = \frac{b}{2d}, \tag{22}$$

where d is the spatial dimension [19–21]. So Eq. (21) can be written as $\tau/\tau_s = 1 + T_c/2(T - T_c)$. Thus, near the critical point one has

$$\tau \sim \frac{1}{T - T_c}. \tag{23}$$

It implies that the system will very slowly approach the equilibrium state, and thus will exhibit the critical slowing down phenomenon. Noting that the correlation length exponent is $\nu = 1/2$ for the Gaussian model [21]; according to the dynamic scaling hypothesis for the relaxation time $\tau \sim \xi^z$ and the scaling behavior of correlation length $\xi \sim |T - T_c|^{-\nu}$ (near the critical point), result (23) gives the dynamic critical exponent z equal to 2. The result is the same as that value of the case of single-spin transition for the Gaussian model with NN interactions [14].

B. Double-spin-transition model

In this section, we study another case, the double-spin-transition model, in which the transitional spins σ_k and σ_j need not be neighbors in spatial distribution, i.e., two spins (σ_k, σ_j) can be arbitrary. Expediently, we still consider the one-dimensional spin system. In this case, for a given σ_k, there are $N-1$ ways to connect with the other spin σ_j, so in the differential equation (8) one has $n = N-1$. The Hamiltonians related to the double spin $(\hat{\sigma}_k, \hat{\sigma}_j)$ are

$$-\beta H_{kj}(\hat{\sigma}_k, \hat{\sigma}_j, \{\sigma_{m\neq k,j}\}) = K[\hat{\sigma}_k(\sigma_{k-1} + \sigma_{k+1}) + \hat{\sigma}_j(\sigma_{j-1} + \sigma_{j+1})] \quad \text{if } j \neq k+1,\ k-1,$$

$$-\beta H_{kj}(\hat{\sigma}_k, \hat{\sigma}_j, \{\sigma_{m\neq k,j}\}) = K(\hat{\sigma}_k \sigma_{k-1} + \hat{\sigma}_k \hat{\sigma}_{k+1} + \hat{\sigma}_{k+1} \sigma_{k+2}) \quad \text{if } j = k+1,$$

and

$$-\beta H_{kj}(\hat{\sigma}_k,\hat{\sigma}_j,\{\sigma_{m\neq k,j}\})=K(\hat{\sigma}_k\sigma_{k+1}+\hat{\sigma}_k\hat{\sigma}_{k-1}+\hat{\sigma}_{k-1}\sigma_{k-2})\quad \text{if}\ \ j=k-1.$$

When k and j are not neighboring, i.e., $j\neq k+1$ and $k-1$, by calculation we can easily obtain

$$\tau_s\sum_{\hat{\sigma}_k,\hat{\sigma}_j}\hat{\sigma}_k W(\sigma_k\sigma_j\to\hat{\sigma}_k\hat{\sigma}_j)=\frac{K(\sigma_{k-1}+\sigma_{k+1})}{b}. \tag{24}$$

For the cases $j=k+1$ and $k-1$, we have obtained the summations as in Eqs. (15) and (16), respectively.

From Eqs. (8), (15), (16), and (24) one obtains

$$\tau_s\frac{d}{dt}s_k(t)=-s_k(t)+\frac{\tau_s}{2t}[x_1(s_{k-1}+s_{k+1})+x_2(s_{k-2}+s_{k+2})] \tag{25}$$

with

$$x_1=\frac{2(N-2)Kb^2-2(N-3)K^3}{(N-1)b(b^2-K^2)}\,\frac{t}{\tau_s},$$

$$x_2=\frac{2bK^2}{(N-1)b(b^2-K^2)}\,\frac{t}{\tau_s}. \tag{26}$$

Comparing Eq. (25) with Eq. (17) we can directly write the solution of Eq. (25) as Eq. (18). So we can get the relaxation time $\tau=\tau_s/\gamma$, where

$$\gamma=\frac{(b-2K)[(N-1)b-(N-3)K]}{(N-1)b(b-K)}.$$

Here, we find that the relaxation time is related to the number of spins of the system. In the case of the thermodynamic limit $N\to\infty$, we can obtain that $\tau=[b/(b-2K)]\tau_s$. If we notice Eq. (22) again, one has $\tau\sim(T-T_c)^{-1}$ near the critical point. Further, one gets that the dynamical critical exponent equals 2 as well.

If we look into such a case, in which there is not only a single-spin transition but also double-spin transitions at one time, we can find that the value of z does not change. It is to say that the admixture of single-spin and double-spin transitions does not change the dynamical critical exponent.

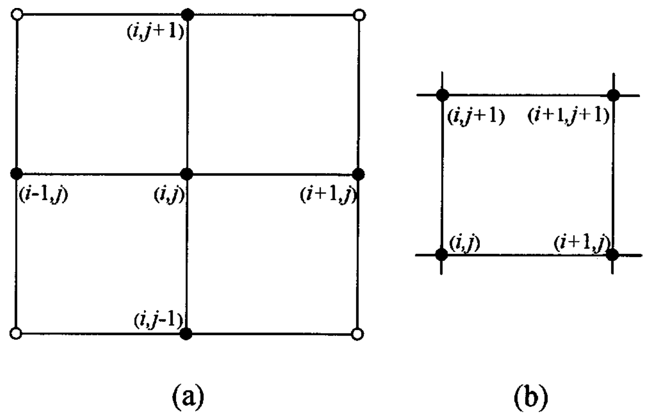

FIG. 1. The sketches of the cluster in the two-dimensional lattice: (a) five-spin cluster and (b) four-spin cluster.

IV. TWO-DIMENSIONAL CASE

In this section, we study the two-dimensional kinetic Gaussian spin system with spin-cluster transitions. Theoretically, the shape of cluster can be arbitrarily chosen, but the calculation is very complex, in general. In order to study the influence of the different clusters on the dynamic critical phenomena, we chose two kinds of clusters: the one named the five-spin cluster is formed from a spin and its NN spins, which is plotted in Fig. 1(a), and the other named the four-spin cluster is formed from four spins [see Fig. 1(b)].

A. Five-spin cluster

First, we study the case of five-spin cluster. Consider an arbitrary cluster c_i composed of sites $(i,\ j)$, $(i+1,j)$, $(i-1,j)$, $(i,j+1)$, and $(i,j-1)$ [see Fig. 1(a)]. The spins of the cluster is $\sigma_{c_i}=(\sigma_{i,j},\sigma_{i+1,j},\sigma_{i-1,j},\sigma_{i,j+1},\sigma_{i,j-1})$. In order to find out the dynamic critical exponent, we calculate the time evolution of the local magnetization $s_{k,l}(t)$. Noting that there are five clusters that contain site $(k,\ l)$, i.e., site $(k,\ l)$ belongs to five different clusters (see Fig. 2). According to Eq. (8), one can easily write the following differential equation:

$$\frac{d}{dt}s_{k,l}(t)=-\frac{s_{k,l}(t)}{\tau_s}+\frac{1}{5}\sum_{\sigma}\sum_{i=1}^{5}\sum_{\hat{\sigma}_{c_i}}\hat{\sigma}_{k,l}W(\sigma_{c_i}\to\hat{\sigma}_{c_i})P(\sigma,t), \tag{27}$$

where

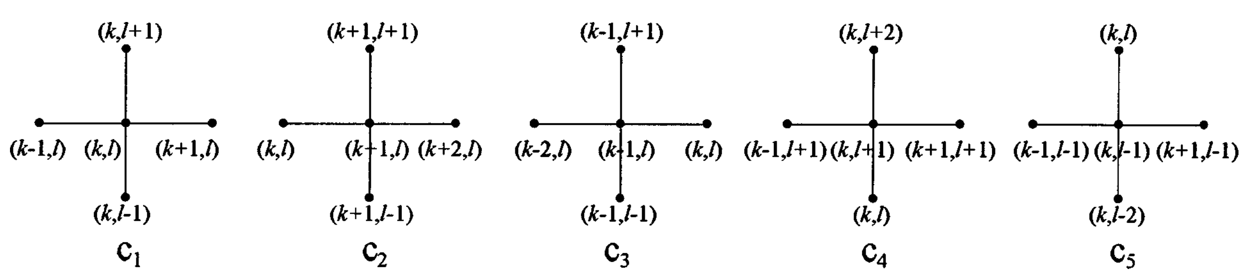

FIG. 2. For the case of the five-spin cluster in the two-dimensional lattice, there are five different clusters that contain site $(k,\ l)$.

$$\sigma_{c_1}=(\sigma_{k,l},\sigma_{k+1,l},\sigma_{k-1,l},\sigma_{k,l+1},\sigma_{k,l-1}),$$

$$\sigma_{c_2}=(\sigma_{k+1,l},\sigma_{k+2,l},\sigma_{k,l},\sigma_{k+1,l+1},\sigma_{k+1,l-1}),$$

$$\sigma_{c_3}=(\sigma_{k-1,l},\sigma_{k,l},\sigma_{k-2,l},\sigma_{k-1,l+1},\sigma_{k-1,l-1}),$$

$$\sigma_{c_4}=(\sigma_{k,l+1},\sigma_{k+1,l+1},\sigma_{k-1,l+1},\sigma_{k,l+2},\sigma_{k,l}),$$

$$\sigma_{c_5}=(\sigma_{k,l-1},\sigma_{k+1,l-1},\sigma_{k-1,l-1},\sigma_{k,l},\sigma_{k,l-2}).$$

Using the same method as in Sec. III, we can also calculate the summations $\Sigma_{\hat{\sigma}_{c_i}}\hat{\sigma}_{k,l}W(\sigma_{c_i}\to\hat{\sigma}_{c_i})$, $i=1,2,...,5$. For the cluster c_1, according to expression (9) the Hamiltonian related to spin cluster $\hat{\sigma}_{c_1}$ is written as

$$\mathcal{H}_{c_1}=K\sigma_{x+}\hat{\sigma}_{k+1,l}+K\sigma_{x-}\hat{\sigma}_{k-1,l}+K\sigma_{y+}\hat{\sigma}_{k,l+1}+K\sigma_{y-}\hat{\sigma}_{k,l-1},\tag{28}$$

where σ_{x+}, σ_{x-}, σ_{y+}, and σ_{y-} are defined as follows:

$$\sigma_{x+}=\hat{\sigma}_{k,l}+\sigma_{k+2,l}+\sigma_{k+1,l+1}+\sigma_{k+1,l-1},$$

$$\sigma_{x-}=\hat{\sigma}_{k,l}+\sigma_{k-2,l}+\sigma_{k-1,l+1}+\sigma_{k-1,l-1},$$

$$\sigma_{y+}=\hat{\sigma}_{k,l}+\sigma_{k,l+2}+\sigma_{k+1,l+1}+\sigma_{k-1,l+1},$$

$$\sigma_{y-}=\hat{\sigma}_{k,l}+\sigma_{k,l-2}+\sigma_{k+1,l-1}+\sigma_{k-1,l-1}.$$

Based on Eqs. (5) and (28), the normalized factor is written as

$$Q_{c_1}=\sum_{\hat{\sigma}_{c_1}}\exp[\mathcal{H}_{c_1}]=\left(\frac{b}{2\pi}\right)^{5/2}\int_{-\infty}^{+\infty}\cdots\times\int_{-\infty}^{+\infty}d\hat{\sigma}_{k,l}d\hat{\sigma}_{k+1,l}d\hat{\sigma}_{k-1,l}d\hat{\sigma}_{k,l+1}d\hat{\sigma}_{k,l-1}\exp[\mathcal{H}_{c_1}]\times\exp\left[-\frac{b}{2}(\sigma_{k,l}^2+\hat{\sigma}_{k+1,l}^2+\hat{\sigma}_{k-1,l}^2+\hat{\sigma}_{k,l+1}^2+\hat{\sigma}_{k,l-1}^2)\right].\tag{29}$$

From Eqs. (4), (28), and (29) we can obtain (see Appendix B)

$$\tau_s\sum_{\hat{\sigma}_{c_1}}\hat{\sigma}_{k,l}W(\sigma_{c_1}\to\hat{\sigma}_{c_1})=\frac{K^2\sigma_{n_1}}{b^2-4K^2},\tag{30}$$

where

$$\sigma_{n_1}=\sigma_{k+2,l}+\sigma_{k-2,l}+\sigma_{k,l+2}+\sigma_{k,l-2}+2\sigma_{k+1,l+1}+2\sigma_{k+1,l-1}+2\sigma_{k-1,l+1}+2\sigma_{k-1,l-1}\tag{31}$$

is the summation of the NN spins of the cluster c_1. For other clusters c_i $(i=2,3,4,5)$, we also get

$$\tau_s\sum_{\hat{\sigma}_{c_i}}\hat{\sigma}_{k,l}W(\sigma_{c_i}\to\hat{\sigma}_{c_i})=\frac{K^2\sigma_{n_i}}{b^2-4K^2},\quad i=2,3,4,5,\tag{32}$$

where σ_{n_i} is the summation of NN spins of the cluster c_i. Thus, from Eqs. (27), (30), and (32) one has

$$\tau_s\frac{d}{dt}s_{k,l}(t)=-s_{k,l}(t)+\frac{1}{5}\frac{K^2}{b^2-4K^2}\sum_{i=1}^{5}s_{n_i}(t),\tag{33}$$

with

$$s_{n_1}=s_{k+2,l}+s_{k-2,l}+s_{k,l+2}+s_{k,l-2}+2s_{k+1,l+1}+2s_{k+1,l-1}+2s_{k-1,l+1}+2s_{k-1,l-1},$$

$$s_{n_2}=s_{k+3,l}+s_{k-1,l}+s_{k+1,l+2}+s_{k+1,l-2}+2s_{k+2,l+1}+2s_{k+2,l-1}+2s_{k,l+1}+2s_{k,l-1},$$

$$s_{n_3}=s_{k-3,l}+s_{k+1,l}+s_{k-1,l+2}+s_{k-1,l-2}+2s_{k-2,l+1}+2s_{k-2,l-1}+2s_{k,l+1}+2s_{k,l-1},$$

$$s_{n_4}=s_{k,l+3}+s_{k,l-1}+s_{k+2,l+1}+s_{k-2,l+1}+2s_{k+1,l+2}+2s_{k-1,l+2}+2s_{k+1,l}+2s_{k-1,l},$$

$$s_{n_5}=s_{k,l-3}+s_{k,l+1}+s_{k+2,l-1}+s_{k-2,l-1}+2s_{k+1,l-2}+2s_{k-1,l-2}+2s_{k+1,l}+2s_{k-1,l}.$$

Equation (33) can be exactly solved by defining a generating function as well (see Appendix C). The solution is

$$s_{k,l}(t)=e^{-t/\tau_s}\sum_{\{m_i=-\infty\}_{i=1,12}}^{\infty}s_{k-p,l-q}(0)I_{m_1}(x)I_{m_2}(x)I_{m_3}(x)I_{m_4}(x)I_{m_5}(5x)I_{m_6}(5x)\times I_{m_7}(3x)I_{m_8}(3x)I_{m_9}(3x)I_{m_{10}}(3x)I_{m_{11}}(2x)I_{m_{12}}(2x);\tag{34}$$

where

$$p=3m_1+2m_3+m_5+m_7+m_8+2m_9-m_{10}+m_{11}+m_{12},$$

$$q=3m_2+2m_4+m_6+2m_7-m_8+m_9+m_{10}+m_{11}-m_{12},$$

$$x=\frac{2K^2}{5(b^2-4K^2)}\frac{t}{\tau_s}.$$

When $t\to\infty$, solution (34) is asymptotically written as

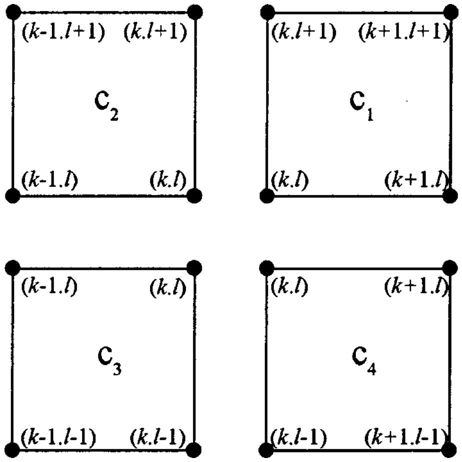

FIG. 3. For the case of the four-spin cluster, there are four different clusters that contain site (k, l).

$$s_{k,l}(t) \sim \frac{\tau_s^6}{t^6\left(\frac{K^2}{b^2-4K^2}\right)^6} \sum_{\{m_i=-\infty\}_{i=1,12}}^{\infty} s_{k-p,l-q}(0) e^{-t/\tau} \sim \left(\frac{\tau_s}{t}\right)^6 e^{-t/\tau},$$

where $s_{k-p,l-q}(0)$ is the initial value of $s_{k-p,l-q}(t)$, and

$$\tau = \frac{b^2-4K^2}{b^2-16K^2}\tau_s . \tag{35}$$

Noting Eq. (22) and $K+K_c \approx 2K_c$ near the critical point, one then gets

$$\tau = \left(1+\frac{3}{8}\frac{T_c}{T-T_c}\right)\tau_s \sim (T-T_c)^{-1}. \tag{36}$$

Therefore, the dynamic critical exponent also equals 2.

B. Four-spin transitions

Consider a four-spin-transition case. We chose the cluster c_i with the sites (i, j), ($i+1$,j), (i,$j+1$), and ($i+1$,$j+1$) [see Fig. 1(b)]. The corresponding spin cluster is denoted by $\sigma_{c_i} = (\sigma_{i,j}, \sigma_{i+1,j}, \sigma_{i,j+1}, \sigma_{i+1,j+1})$. We also focus on the differential equation (8). From Fig. 3 we can see that there are four-spin clusters that contain $\sigma_{k,l}$ (see Fig. 3):

$$\sigma_{c_1} = (\sigma_{k,l}, \sigma_{k+1,l}, \sigma_{k,l+1}, \sigma_{k+1,l+1}),$$

$$\sigma_{c_2} = (\sigma_{k,l}, \sigma_{k+1,l}, \sigma_{k,l-1}, \sigma_{k+1,l-1}),$$

$$\sigma_{c_3} = (\sigma_{k,l}, \sigma_{k-1,l}, \sigma_{k,l-1}, \sigma_{k-1,l-1}),$$

$$\sigma_{c_4} = (\sigma_{k,l}, \sigma_{k-1,l}, \sigma_{k,l+1}, \sigma_{k-1,l+1}).$$

Following the calculation in Sec. III, we can get the differential equation as (see Appendix D)

$$\frac{d}{dt}s_{k,l}(t) = -\frac{1}{\tau_s}s_{k,l}(t) + \frac{1}{\tau_s}\frac{K(b^2-2K^2)s_{n_1}(t)+K^2 b s_{n_2}(t)+K^3 s_{n_3}(t)}{2(b^3-4bK^2)}, \tag{37}$$

with

$$s_{n_1} = s_{k-1,l}+s_{k+1,l}+s_{k,l-1}+s_{k,l+1},$$

$$s_{n_2} = s_{k-1,l+1}+s_{k+1,l+1}+s_{k,l+2}+s_{k,l-2}+s_{k+1,l-1} + s_{k-1,l-1}+s_{k+2,l}+s_{k-2,l},$$

and

$$s_{n_3} = s_{k+1,l+2}+s_{k-1,l+2}+s_{k+2,l+1}+s_{k-2,l+1}+s_{k+1,l-2} + s_{k-1,l-2}+s_{k+2,l-1}+s_{k-2,l-1}.$$

Equation (37) can be exactly solved. Following Appendix C, one gets

$$s_{k,l}(t) = e^{-t/\tau_s}\sum_{\{m_i=-\infty\}_{i=1.10}}^{\infty} s_{k-p,l-q}(0) I_{m_1}(x_1) I_{m_2}(x_1) \times I_{m_3}(x_2) I_{m_4}(x_2) I_{m_5}(x_2) \times I_{m_6}(x_2) I_{m_7}(x_3) I_{m_8}(x_3) I_{m_9}(x_3) I_{m_{10}}(x_3), \tag{38}$$

with

$$p = m_1+m_3+m_4+2m_5+m_7+2m_8-m_9+m_{10},$$

$$q = m_2+m_3+m_4+2m_6+2m_7+m_8+m_9-m_{10},$$

$$x_1 = \frac{K(b^2-2K^2)}{b^3-4bK^2}\frac{t}{\tau_s}, \quad x_2 = \frac{bK^2}{b^3-4bK^2}\frac{t}{\tau_s},$$

$$x_3 = \frac{K^3}{b^3-4bK^2}\frac{t}{\tau_s}.$$

When $t \to \infty$, Eq. (38) is asymptotically expressed as $s_{k,l}(t) \sim e^{-t/\tau}$, in which the relaxation time is given as

$$\tau = \frac{b-2K}{b-4K}\tau_s \sim (T-T_c)^{-1}. \tag{39}$$

Compare Eqs. (39) with (36), although the forms of the two relaxation times are different, the dynamic critical behavior of the two systems is the same near the critical point. Thus, the value of the dynamic critical exponent z equals 2 as well.

V. THREE-DIMENSIONAL CASE

The three-dimensional kinetic Gaussian model can also be exactly solved. Considering both the simplification of calculation and typicalness of the spin cluster, we study the case of the seven-spin cluster transitions, which is similar to the

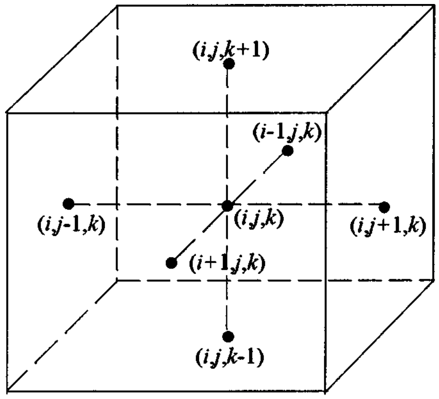

FIG. 4. The sketch of the seven-spin cluster in the three-dimensional lattice.

case of the five-spin cluster in Sec. IV. The cluster considered here contains the site (i, j, k) and its NN sites $(i+1,j,k)$, $(i-1,j,k)$, $(i,j+1,k)$, $(i,j-1,k)$, $(i,j,k+1)$, and $(i,j,k-1)$ (see Fig. 4). In order to solve the differential equation of the local magnetization $s_{k,l,n}(t)$, we need to write the spin clusters that contain spin $\sigma_{k,l,n}$. There are seven such clusters:

$$\sigma_{c_1}=(\sigma_{k,l,n},\sigma^{nn}_{k,l,n}),\quad \sigma_{c_2}=(\sigma_{k+1,l,n},\sigma^{nn}_{k+1,l,n}),$$

$$\sigma_{c_3}=(\sigma_{k-1,l,n},\sigma^{nn}_{k-1,l,n}),$$

$$\sigma_{c_4}=(\sigma_{k,l+1,n},\sigma^{nn}_{k,l+1,n}),\quad \sigma_{c_5}=(\sigma_{k,l-1,n},\sigma^{nn}_{k,l-1,n}),$$

$$\sigma_{c_6}=(\sigma_{k,l,n+1},\sigma^{nn}_{k,l,n+1}),$$

$$\sigma_{c_7}=(\sigma_{k,l,n-1},\sigma^{nn}_{k,l,n-1}),$$

where $\sigma^{nn}_{k,l,n}$, $\sigma^{nn}_{k+1,l,n},\ldots$ denote the sets of the NN spins of $\sigma_{k,l,n}$, $\sigma_{k+1,l,n},\ldots$, respectively, e.g., $\sigma^{nn}_{k,l,n}=(\sigma_{k+1,l,n},\sigma_{k-1,l,n},\sigma_{k,l+1,n},\sigma_{k,l-1,n},\sigma_{k,l,n+1},\sigma_{k,l,n-1})$. Based on Eq. (8), the differential equation of the local magnetization $s_{k,l,n}(t)$ is written as

$$\frac{d}{dt}s_{k,l,n}(t)=-\frac{s_{k,l,n}(t)}{\tau_s}+\frac{1}{7}\sum_{\sigma}\sum_{i=1}^{7}\sum_{\hat{\sigma}_{c_i}}\hat{\sigma}_{k,l,n}W(\sigma_{c_i}\rightarrow\hat{\sigma}_{c_i})P(\sigma,t). \tag{40}$$

Using the same method as in Sec. III, we can also calculate the summations $\Sigma_{\hat{\sigma}_{c_i}}\hat{\sigma}_{k,l}W(\sigma_{c_i}\rightarrow\hat{\sigma}_{c_i})$, $i=1,2,\ldots,7$. For the cluster c_1, the Hamiltonian related to spin cluster $\hat{\sigma}_{c_1}$ is written as

$$\mathcal{H}_{c_1}=K(\sigma_{x+}\hat{\sigma}_{k+1,l,n}+\sigma_{x-}\hat{\sigma}_{k-1,l,n}+\sigma_{y+}\hat{\sigma}_{k,l+1,n}+\sigma_{y-}\hat{\sigma}_{k,l-1,n}+\sigma_{z+}\hat{\sigma}_{k,l,n+1}+\sigma_{z-}\hat{\sigma}_{k,l,n-1}), \tag{41}$$

where σ_{x+}, σ_{x-}, σ_{y+}, σ_{y-}, σ_{z+}, and σ_{z-} are the summations of the NN spins of $\hat{\sigma}_{k+1,l,n}$, $\hat{\sigma}_{k-1,l,n}$, $\hat{\sigma}_{k,l+1,n}$, $\hat{\sigma}_{k,l-1,n}$, $\hat{\sigma}_{k,l,n+1}$, and $\hat{\sigma}_{k,l,n-1}$, respectively, e.g.,

$$\sigma_{x+}=\hat{\sigma}_{k,l,n}+\sigma_{k+2,l,n}+\sigma_{k+1,l+1,n}+\sigma_{k+1,l-1,n}+\sigma_{k+1,l,n+1}+\sigma_{k+1,l,n-1},$$

$$\sigma_{x-}=\hat{\sigma}_{k,l,n}+\sigma_{k-2,l,n}+\sigma_{k-1,l+1,n}+\sigma_{k-1,l-1,n}+\sigma_{k-1,l,n+1}+\sigma_{k-1,l,n-1}.$$

From Eqs. (4), (2), and (41) we can get

$$\tau_s\sum_{\hat{\sigma}_{c_1}}\hat{\sigma}_{k,l,n}W(\sigma_{c_1}\rightarrow\hat{\sigma}_{c_1})=\frac{K^2\sigma_{n_1}}{b^2-6K^2}, \tag{42}$$

in which σ_{n_1} is the summation of NN spins of the cluster c_1, given as

$$\begin{aligned}\sigma_{n_1}=&\sigma_{k+2,l,n}+\sigma_{k-2,l,n}+\sigma_{k,l+2,n}+\sigma_{k,l-2,n}+\sigma_{k,l,n+2}\\&+\sigma_{k,l,n-2}+2\sigma_{k+1,l+1,n}+2\sigma_{k+1,l-1,n}+2\sigma_{k+1,l,n+1}\\&+2\sigma_{k+1,l,n-1}+2\sigma_{k-1,l+1,n}+2\sigma_{k-1,l-1,n}\\&+2\sigma_{k-1,l,n+1}+2\sigma_{k-1,l,n-1}+2\sigma_{k,l+1,n+1}\\&+2\sigma_{k,l+1,n-1}+2\sigma_{k,l-1,n+1}+2\sigma_{k,l-1,n-1}.\end{aligned}$$

For other clusters c_i, the summations $\Sigma_{\hat{\sigma}_{c_i}}\hat{\sigma}_{k,l,n}W(\sigma_{c_i}\rightarrow\hat{\sigma}_{c_i})$ ($i=2,\ldots,7$) can be obtained as well. Thus, from Eqs. (40) and (42) one has

$$\tau_s\frac{d}{dt}s_{k,l,n}(t)=-s_{k,l,n}(t)+\frac{1}{7}\frac{K^2}{b^2-6K^2}\sum_{i=1}^{7}s_{n_i}(t), \tag{43}$$

where $s_{n_i}(t)=\Sigma_\sigma\sigma_{n_i}P(\sigma,t)$. This equation contains many terms, and its solving process and solution are very complex. If we only regard the relaxation time of the system, we can focus on the total magnetization

$$m(t)=\sum_{(k,l,n)}s_{k,l,n}(t), \tag{44}$$

where the summation $\Sigma_{(k,l,n)}$ goes over all sites in the system. Performing a summation for site (k,l,n) in Eq. (43) one can easily obtain the magnetization $m(t)$ as

$$m(t)=m(0)e^{-t/\tau}, \tag{45}$$

where $m(0)=\Sigma_{(k,l,n)}s_{k,l,n}(0)$ is the initial value of the total magnetization and the relaxation time is

$$\tau=\frac{b^2-6K^2}{b^2-36K^2}\tau_s. \tag{46}$$

Noting Eq. (22), we can see that the relaxation time for T near T_c will also approach ∞, and that it can be written as $\tau=(1+\frac{5}{12}T_c/(T-T_c))\tau_s$. Further, this result can give the same value of the dynamic critical exponent as above.

According to results (35) and (46), for the d-dimensional kinetic Gaussian model, one can obtain the general form of the relaxation time

$$\tau = \frac{b^2 - 2dK^2}{b^2 - 4d^2K^2} \tau_s . \tag{47}$$

Near the critical point, $\tau \approx 1 + [(2d-1)T_c/4d(T-T_c)]\tau_s \sim (T-T_c)^{-1}$, thus there $z=2$.

VI. CONCLUSION AND DISCUSSION

In this paper, we presented a multispin-transition model and studied the dynamic critical phenomena of the Gaussian model with NN interactions. The time evolution of the magnetizations is investigated and the exact results of the relaxation time and the dynamical critical exponent are obtained. Our results show that the dynamical critical exponent is independent of the choice of the transitional spins, namely, no matter whether the transitional spins are concentrated (Secs. III A, IV, and V) or distributed (Sec. III B), no matter whether the transitional spins have different spatial configurations [Figs. 1(a) and (b)], the dynamical critical exponent is the same. Combining it with Glauber's single transition mechanism, we lead to the following conclusion: the dynamical critical exponent is independent of the transition mechanism, in general.

For the above conclusion, we may interpret the following: from the view of renormalization a transitional spin cluster can be transformed into a single spin through a renormalization transformation (scaling transformation), and thus the behavior of a spin cluster is essentially equal to that of a single spin. Therefore, different transition mechanisms give the same value of the dynamical critical exponent.

Our results also show that the value of the dynamical critical exponent is the same for one-, two-, and three-dimensional Gaussian models with NN interactions. At first sight, it seems to be in conflict with the recent view of universality class. In fact, it can be regarded as a special accidental degeneracy phenomena, which is just like $\nu = 1/2$ for any dimensional Gaussian model [20,21].

It is necessary to point out that the spin clusters considered in this paper are all small, nonpercolating ones, and that the transitions of percolating spin clusters, as in the case of the Swendsen-Wang or Wolff cluster algorithms [22,23], are not considered. Our results only hold for the former. When transitions of percolating spin clusters are considered, the results may be changed.

Our method can be generalized for other models, in principle, such as the Ising model in two or three dimensions, but the calculations will be very complex, so that it is difficult to obtain the exact analytical results.

Finally, our multispin transition mechanism may include Kawasaki-like dynamics, if we ask for a conservation of the total value of all spins in the transitional spin cluster.

ACKNOWLEDGMENTS

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APPENDIX A: SOLVE EQ. (17)

Let us define a generating function

$$F(\lambda,t) = \sum_{k=-\infty}^{\infty} \lambda^k s_k(t). \tag{A1}$$

Based on Eq. (17), we have

$$\frac{\partial F(\lambda,t)}{\partial t} = \sum_{k=-\infty}^{\infty} \lambda^k \frac{d}{dt} s_k(t) = -\frac{1}{\tau_s}\left[1 - \frac{K[b(\lambda+\lambda^{-1}) + K(\lambda^2+\lambda^{-2})]}{2(b^2-K^2)}\right] F(\lambda,t).$$

The solution of the above equation is

$$F(\lambda,t) = F(\lambda,0) e^{-t/\tau_s} \exp\left[\frac{Kbt}{2(b^2-K^2)}(\lambda+\lambda^{-1})\right] \times \exp\left[\frac{K^2 t}{2(b^2-K^2)}(\lambda^2+\lambda^{-2})\right]. \tag{A2}$$

Noting the generating function of the Bessel function of imaginary argument [1],

$$\exp\left(\frac{x}{2}(\lambda+\lambda^{-1})\right) = \sum_{\nu=-\infty}^{+\infty} \lambda^{\nu} I_{\nu}(x), \tag{A3}$$

we obtain that

$$F(\lambda,t) = F(\lambda,0) e^{-t/\tau_s} \sum_{m,n=-\infty}^{\infty} \lambda^{m+2n} I_m(x_1) I_n(x_2), \tag{A4}$$

where both x_1 and x_2 are determined by Eq. (19). From Eqs. (A1) and (A4), we get

$$\sum_{k=-\infty}^{\infty} \lambda^k s_k(t) = e^{-t/\tau_s} \sum_{l=-\infty}^{\infty} \sum_{m,n=-\infty}^{\infty} \times s_l(0) \lambda^{m+2n+l} I_m(x_1) I_n(x_2).$$

Comparing the two sides of the above equation, one obtains solution (18).

APPENDIX B: CALCULATION OF EXPRESSION (30)

For the two-dimensional Gaussian system, from Eqs. (28) and (29) we have

$$Q_{c_1}=\sum_{\hat{\sigma}_{c_1}} \exp[\mathcal{H}_{c_1}]=\left(\frac{b}{2\pi}\right)^{5/2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\hat{\sigma}_{k,l}d\hat{\sigma}_{k+1,l}d\hat{\sigma}_{k-1,l}d\sigma_{k,l+1}d\hat{\sigma}_{k,l-1}\exp\left(-\frac{b}{2}\sigma_{k,l}^2\right)$$
$$\times\exp\left[K\sigma_{x+}\hat{\sigma}_{k+1,l}-\frac{b}{2}\hat{\sigma}_{k+1,l}^2\right]\exp\left[K\sigma_{x-}\hat{\sigma}_{k-1,l}-\frac{b}{2}\hat{\sigma}_{k-1,l}^2\right]\exp\left[K\sigma_{y+}\hat{\sigma}_{k,l+1}-\frac{b}{2}\hat{\sigma}_{k,l+1}^2\right]\exp\left[K\sigma_{y-}\hat{\sigma}_{k,l-1}-\frac{b}{2}\hat{\sigma}_{k,l-1}^2\right]$$
$$=\sqrt{\frac{b}{2\pi}}\int_{-\infty}^{+\infty} d\hat{\sigma}_{k,l}\exp\left(-\frac{b}{2}\hat{\sigma}_{k,l}^2\right)\exp\left[\frac{K^2}{2b}(\sigma_{x+}^2+\sigma_{x-}^2+\sigma_{y+}^2+\sigma_{y-}^2)\right]$$
$$=\sqrt{\frac{b}{2\pi}}\exp\left(\frac{K^2}{2b}(\sigma_{k+2,l}+\sigma_{k+1,l+1}+\sigma_{k+1,l-1})^2+\frac{K^2}{2b}(\sigma_{k-2,l}+\sigma_{k-1,l+1}+\sigma_{k-1,l-1})^2\right)$$
$$\times\exp\left(\frac{K^2}{2b}(\sigma_{k,l+2}+\sigma_{k+1,l+1}+\sigma_{k-1,l+1})^2+\frac{K^2}{2b}(\sigma_{k,l-2}+\sigma_{k+1,l-1}+\sigma_{k-1,l-1})^2\right)\int_{-\infty}^{+\infty} d\hat{\sigma}_{k,l}$$
$$\times\exp\left[-\frac{b}{2}\left(1-\frac{4K^2}{b^2}\right)\hat{\sigma}_{k,l}^2+\frac{K^2}{b}\sigma_{n_1}\hat{\sigma}_{k,l}\right],$$

where σ_{n_1} has been given in Eq. (31). Thus, one has

$$\tau_s\sum_{\hat{\sigma}_{c_1}}\hat{\sigma}_{k,l}W(\sigma_{c_1}\to\hat{\sigma}_{c_1})=\frac{\Sigma_{\hat{\sigma}_{c_1}}\hat{\sigma}_{k,l}\exp[\mathcal{H}_{c_1}]}{\Sigma_{\hat{\sigma}_{c_1}}\exp[\mathcal{H}_{c_1}]}$$
$$=\frac{\int_{-\infty}^{+\infty}\hat{\sigma}_{k,l}d\hat{\sigma}_{k,l}\exp\left[-\frac{b}{2}\left(1-\frac{4K^2}{b^2}\right)\hat{\sigma}_{k,l}^2+\frac{K^2}{b}\sigma_{n_1}\hat{\sigma}_{k,l}\right]}{\int_{-\infty}^{+\infty}d\hat{\sigma}_{k,l}\exp\left[-\frac{b}{2}\left(1-\frac{4K^2}{b^2}\right)\hat{\sigma}_{k,l}^2+\frac{K^2}{b}\sigma_{n_1}\hat{\sigma}_{k,l}\right]}=\frac{K^2\sigma_{n_1}}{b^2-4K^2}.$$

APPENDIX C: THE SOLUTION OF EQ. (33)

Let

$$F(\lambda_1,\lambda_2,t)=\sum_{k,l=-\infty}^{\infty}\lambda_1^k\lambda_2^l s_{k,l}(t), \tag{C1}$$

from Eq. (33) we get

$$\tau_s\frac{\partial F(\lambda_1,\lambda_2,t)}{\partial t}=-F(\lambda_1,\lambda_2,t)+G(\lambda_1,\lambda_2)F(\lambda_1,\lambda_2,t), \tag{C2}$$

where

$$G(\lambda_1,\lambda_2)=\frac{K^2}{5(b^2-2dK^2)}[(\lambda_1^3+\lambda_1^{-3})+(\lambda_2^3+\lambda_2^{-3})+(\lambda_1^2+\lambda_1^{-2})+(\lambda_2^2+\lambda_2^{-2})+5(\lambda_1+\lambda_1^{-1})+5(\lambda_2+\lambda_2^{-1})+3((\lambda_1\lambda_2^2)+\lambda_1^{-1}\lambda_2^{-2})+3(\lambda_1\lambda_2^{-2}+\lambda_1^{-1}\lambda_2^2)+3((\lambda_1^2\lambda_2+\lambda_1^{-2}\lambda_2^{-1}))+3(\lambda_1^2\lambda_2^{-1}+\lambda_1^{-2}\lambda_2)+2(\lambda_1\lambda_2+\lambda_1^{-1}\lambda_2^{-1})+2(\lambda_1\lambda_2^{-1}+\lambda_1^{-1}\lambda_2)].$$

The solution of Eq. (C2) is

$$F(\lambda_1,\lambda_2,t)=F(\lambda_1,\lambda_2,0)e^{-t/\tau_s}\exp[G(\lambda_1,\lambda_2)t]. \tag{C3}$$

Noting the generating function of the Bessel function (A3) and

$$F(\lambda_1,\lambda_2,0)=\sum_{k',l'=-\infty}^{\infty}\lambda_1^{k'}\lambda_2^{l'}s_{k,l}(0),$$

from Eqs. (C1) and (C3), we can write the following equation:

$$\sum_{k,l=-\infty}^{\infty}\lambda_1^k\lambda_2^l s_{k,l}(t)=e^{-t/\tau_s}\sum_{k',l'=-\infty}^{\infty}s_{k',l'}(0)\times\sum_{\{m_i=-\infty\}_{i=1,12}}^{\infty}\lambda_1^{k'+p}\lambda_2^{l'+q}I_{m_1}(x)\times I_{m_2}(x)I_{m_3}(x)I_{m_4}(x)I_{m_5}(5x)\times I_{m_6}(5x)I_{m_7}(3x)I_{m_8}(3x)I_{m_9}(3x)\times I_{m_{10}}(3x)I_{m_{11}}(2x)I_{m_{12}}(2x).$$

Further, we can get Eq. (34).

APPENDIX D: CALCULATION OF EQ. (37)

The Hamiltonian related to the spin cluster $\hat{\sigma}_{c_1}$ is

$$\mathcal{H}_{c_1}=-\beta H_{c_1}(\hat{\sigma}_{c_1},\{\sigma_{j\notin c_1}\})=K(\sigma_{k-1,l}+\sigma_{k,l-1})\hat{\sigma}_{k,l}+K(\sigma_{k+1,l+2}+\sigma_{k+2,l+1})\hat{\sigma}_{k+1,l+1}+K(\hat{\sigma}_{k,l}+\hat{\sigma}_{k+1,l+1}+\sigma_{k-1,l+1}+\sigma_{k,l+2})\hat{\sigma}_{k,l+1}+K(\hat{\sigma}_{k,l}+\hat{\sigma}_{k+1,l+1}+\sigma_{k+1,l-1}+\sigma_{k+2,l})\hat{\sigma}_{k+1,l}. \quad \text{(D1)}$$

Based on Eq. (D1) the normalized factor Q_{c_1} and $\Sigma_{\hat{\sigma}_{c_1}}\hat{\sigma}_{k,l}W(\sigma_{c_1}\to\hat{\sigma}_{c_1})$ can be calculated. This process is complex, so here we only give the main procedure and results.

$$Q_{c_1}=\sum_{\hat{\sigma}_{c_1}}\exp[-\beta H(\hat{\sigma}_{c_1},\{\sigma_{j\notin c_1}\})]$$
$$=\left(\frac{b}{2\pi}\right)^2\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}d\hat{\sigma}_{k,l}d\hat{\sigma}_{k+1,l}d\hat{\sigma}_{k,l+1}d\hat{\sigma}_{k+1,l+1}\times\exp[-\beta H(\hat{\sigma}_{c_1},\{\sigma_{j\notin c_1}\})]\exp\left[-\frac{b}{2}\hat{\sigma}_{k,l}^2-\frac{b}{2}\hat{\sigma}_{k+1,l+1}^2-\frac{b}{2}\hat{\sigma}_{k+1,l}^2-\frac{b}{2}\hat{\sigma}_{k+1,l}^2\right]=\frac{b}{2\pi}\exp\left(\frac{\alpha_1^2}{2b'}+\frac{\delta}{2b}\right)$$
$$\times\int_{-\infty}^{+\infty}d\hat{\sigma}_{k,l}\exp\left[\left(\alpha_2+\frac{2\alpha_1K^2}{bb'}\right)\hat{\sigma}_{k,l}\right]\exp\left[-\frac{\lambda}{2}\hat{\sigma}_{k,l}^2\right];$$

with

$$\lambda=b\left(1-\frac{2K^2}{b^2}-\frac{4K^4}{b^3b'}\right),$$

$$\alpha_1=\frac{K^2(\sigma_{k-1,l+1}+\sigma_{k,l+2}+\sigma_{k+1,l-1}+\sigma_{k+2,l})}{b}+K(\sigma_{k+1,l+2}+\sigma_{k+2,l+1}),$$

$$\alpha_2=\frac{K^2(\sigma_{k-1,l+1}+\sigma_{k,l+2}+\sigma_{k+1,l-1}+\sigma_{k+2,l})}{b}+K(\sigma_{k-1,l}+\sigma_{k,l-1}),$$

$$\delta=K^2(\sigma_{k-1,l+1}+\sigma_{k,l+2})^2+K^2(\sigma_{k+1,l-1}+\sigma_{k+2,l})^2,$$

$$bb'=b^2-2K^2.$$

$$\tau_s\sum_{\hat{\sigma}_{c_1}}\hat{\sigma}_{k,l}W(\sigma_{c_1}\to\hat{\sigma}_{c_1})=\frac{1}{Q_{c_1}}\exp[-\beta H(\hat{\sigma}_{c_1},\{\sigma_{j\notin c_1}\})]$$
$$=\frac{K(b^2-2K^2)(\sigma_{k-1,l}+\sigma_{k,l-1})+K^2b(\sigma_{k-1,l+1}+\sigma_{k,l+2}+\sigma_{k+1,l-1}+\sigma_{k+2,l})}{b^3-4bK^2}+\frac{2K^3(\sigma_{k+1,l+2}+\sigma_{k+2,l+1})}{b^3-4bK^2}. \quad \text{(D2)}$$

Also, we can obtain

$$\sum_{\hat{\sigma}_{c_2}}\hat{\sigma}_{k,l}W(\sigma_{c_2}\to\hat{\sigma}_{c_2})=\frac{K(b^2-2K^2)(\sigma_{k-1,l}+\sigma_{k,l+1})+K^2b(\sigma_{k-1,l-1}+\sigma_{k,l-2}+\sigma_{k+1,l+1}+\sigma_{k+2,l})}{b^3-4bK^2}+\frac{2K^3(\sigma_{k+1,l-2}+\sigma_{k+2,l-1})}{b^3-4bK^2}, \quad \text{(D3)}$$

$$\sum_{\hat{\sigma}_{c_3}}\hat{\sigma}_{k,l}W(\sigma_{c_3}\to\hat{\sigma}_{c_3})=\frac{K(b^2-2K^2)(\sigma_{k+1,l}+\sigma_{k,l+1})+K^2b(\sigma_{k+1,l-1}+\sigma_{k,l-2}+\sigma_{k-1,l+1}+\sigma_{k-2,l})}{b^3-4bK^2}+\frac{2K^3(\sigma_{k-1,l-2}+\sigma_{k-2,l-1})}{b^3-4bK^2}, \quad \text{(D4)}$$

and

$$\sum_{\hat{\sigma}_{c_4}} \hat{\sigma}_{k,l} W(\sigma_{c_4} \to \hat{\sigma}_{c_4}) = \frac{K(b^2-2K^2)(\sigma_{k+1,l}+\sigma_{k,l-1})+K^2 b(\sigma_{k+1,l+1}+\sigma_{k,l+2}+\sigma_{k-1,l-1}+\sigma_{k-2,l})}{b^3-4bK^2} + \frac{2K^3(\sigma_{k-1,l+2}+\sigma_{k-2,l+1})}{b^3-4bK^2}. \quad \text{(D5)}$$

From above, Eqs. (D2)–(D5), according to Eq. (8), one can obtain the differential equation as in Eq. (37).

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